Matroids and Greedy Algorithms

Monday, 6 September 2021 2:00 PM

Matroids are combinatorial structures that generalize many well-known set systems. They provide a unified theory of why greedy algorithms work.

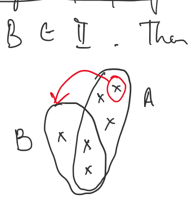
Given: $S = \{s_1, s_2, \dots, s_n\}$

$\mathcal{I} \subseteq 2^S$ is a family of subsets of S (aka a set system)

① $\mathcal{I}$ must be hereditary or downward-closed (if $A \in \mathcal{I}$, and $B \subseteq A$, then $B \in \mathcal{I}$)

(assume throughout $\mathcal{I} \neq \phi$) then clearly $\phi \in \mathcal{I}$

② Exchange property: $A \in \mathcal{I}$, B is s.t. $|B| < |A|$ and $B \in \mathcal{I}$. Then $\exists x \in A \setminus B$ s.t. $B \cup \{x\} \in \mathcal{I}$



If $M = (S, \mathcal{I})$ is a set system that satisfies ① & ⑪, then M is a matroid

Sets in $\mathcal{I}$ are called independent sets

$S = \{a, b, c, d\}$
$\{a, b, c\}, \{b, d\}, \{c, d\} \in \mathcal{I}$
$A = \{a, b, c\}$, $B = \{b, d\}$
$\{a, b\}, \{b, c\}, \{a, c\}$
one of $\{a, b, d\}$ or $\{b, c, d\}$ must be independent

Example:

$S = \{1, 2, 3, 4\}$ $\quad A = \{1, 2, 3\}$, $B = \{4\} \in \mathcal{I}$

Then: $\phi \in \mathcal{I}$, all sets of size 1, $\{1,2\}, \{2,3\}, \{1,3\}$, and at least one of the others $\{1,2,3\}$, and at least one more

$\{1, 2, 3, 4\}$ may not be independent

E.g. $\mathcal{I} = \{123, 12, 23, 13, 412, 41, 42, 4, 1, 2, 3, \phi\}$

Example 1: For any set S of elements $\{s_1, \dots, s_n\}$,

Uniform Matroid: let $\mathcal{I} = \{S' \subseteq S : |S'| \leq k\}$ for some $k \in \mathbb{Z}^+$

Claim: $M = (S, \mathcal{I})$ is a matroid

Proof: ① Downward-closed clearly satisfied because say $X \in \mathcal{I} \Rightarrow |X| \leq k$, then for any $Y \subseteq X$, $|Y| \leq |X|$, hence $Y \in \mathcal{I}$

⑪ Given $X, Y \in \mathcal{I}$, $|X| < |Y|$. Then $|X| \leq k-1$. Take any $a \in Y \setminus X$, then $X \cup \{a\} \in \mathcal{I}$ since $|X \cup \{a\}| \leq k$

Hence $M = (S, \mathcal{I})$ is a matroid

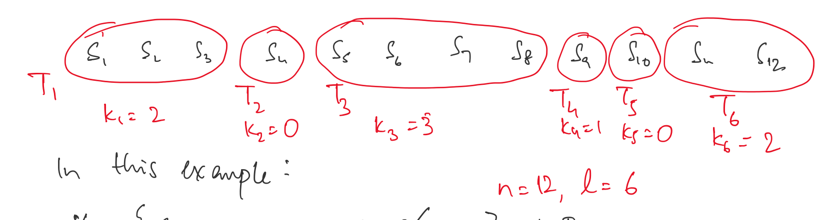
Example 1.5: Partition Matroid

Given $S = \{s_1, s_2, \dots, s_n\}$, $T_1, T_2, \dots, T_\ell \subseteq S$ s.t. T_i's are disjoint and $\bigcup_{i=1}^{\ell} T_i = S$

Also $k_1, k_2, \dots, k_\ell \in \mathbb{Z}_+$.

Set system $(S, \mathcal{I})$, where $\mathcal{I} = \{S' \subseteq S : |S' \cap T_i| \leq k_i$ for $i = 1 \dots \ell\}$

Claim: $M = (S, \mathcal{I})$ is a matroid.



In this example: $n = 12$, $\ell = 6$

$X = \{s_1, s_2, s_4, s_5, s_9, \cancel{s_{10}}, s_{11}\} \notin \mathcal{I}$

Proof: Need to show $(S, \mathcal{I})$ satisfies earlier 2 properties

① Downward-closed: Say $X \in \mathcal{I}$, $Y \subseteq X$

Since $Y \cap T_i \subseteq X \cap T_i$, and hence $|Y \cap T_i| \leq |X \cap T_i| \leq k_i$

⑪ Exchange property: Say $X \in \mathcal{I}$, $Y \in \mathcal{I}$, $|Y| < |X|$

Since $|Y| < |X|$, $\exists i \in \{1, \dots, \ell\}$ s.t. $|Y \cap T_i| < |X \cap T_i| \leq k_i$

Take any elt. $a \in (X \cap T_i) \setminus (Y \cap T_i)$, add a to Y. Then $Y \cup \{a\}$ is also independent. ∎

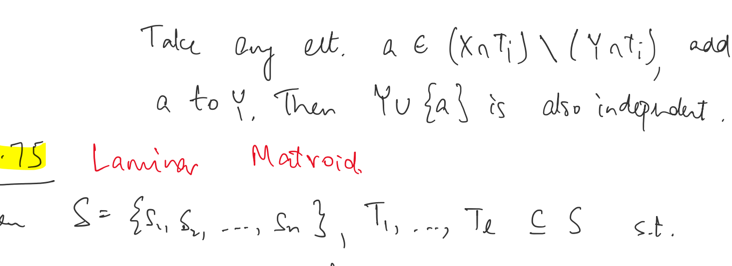
Example 1.75 Laminar Matroid

(find notes/books w/ these examples)

Given $S = \{s_1, s_2, \dots, s_n\}$, $T_1, \dots, T_\ell \subseteq S$ s.t. for all $i, j \in \{1, \dots, \ell\}$ either: ① $T_i \subseteq T_j$, ⑪ $T_j \subseteq T_i$, or ⑪ T_i, T_j are disjoint and $k_1, \dots, k_\ell$

Define $\mathcal{I} = \{S' \subseteq S : |S' \cap T_i| \leq k_i$ for all $i \leq \ell\}$

Example:



Claim: $M = (S, \mathcal{I})$ is a matroid. Prove yourself.

Say $S' = \{s_1, s_2\} \notin \mathcal{I}$ $\quad S' = \{s_4\} \in \mathcal{I}$

Example 2: Given a finite undirected graph $G = (V, E)$,

Consider $(E, \mathcal{I})$ where $\mathcal{I} = \{E' \subseteq E : E'$ is acyclic$\}$.

i.e., all forests in G are independent sets.

Claim: $M = (E, \mathcal{I})$ is a matroid, called graphic matroid.

Do yourself: prove that this is indeed a matroid

Example 3: Given a matrix $A \in \mathbb{R}^{m \times n}$. Now $S = \{1, 2, 3, \dots, n\}$, and $\mathcal{I} = \{S' \subseteq S :$ s.t. the columns with indices in S are linearly independent$\}$

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$
$$\begin{bmatrix} 1 & 1 & 5 & 2 & 3 & 2 \\ 2 & 7 & 7 & 6 & 6 & 9 \\ 5 & 9 & 9 & 4 & 15 & 14 \end{bmatrix}$$

Claim: $M = (S, \mathcal{I})$ is a matroid, Prove yourself

$M = (\{1, 2, 3, 4, 5, 6\}, \mathcal{I})$

This is what is called a linear matroid, or a matric matroid

Q. is $\{1, 2, 6\} \in \mathcal{I}$? No, since columns 1, 2, 6 are not linearly independent

Ok, back to properties.

Fix matroid $M = (S, \mathcal{I})$. Given $X \in \mathcal{I}$, $a \notin X$.

Then a "extends" X if $X \cup \{a\} \in \mathcal{I}$

If X is maximal then no element extends X.

Let X, Y be maximal independent sets. Then $|X| = |Y|$ by exchg. prop.

Therefore, every maximal IS is also a maximum IS.

To find a max IS in the matroid $M = (S, \mathcal{I})$, where $|S| = n$, consider the following algorithm:

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T ← φ
for i = 1 ... n
    if T∪{s_i} ∈ I, add s_i to T
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Check: if a extends X, then a extends any subset of X

Claim: T obtained by the algo is a maximum IS

Proof: Clearly T is independent

Further T is maximal since if s_i extends T, it should extend any subset of T also, and hence the algo should have added s_i to T. ∎

Now suppose each element s_i has a nonnegative weight $w(s_i)$. Problem: find IS of maximum weight.

Algorithm: assume $w(s_1) \geq w(s_2) \geq \dots \geq w(s_n)$

```
T ← φ
for i = 1 ... n
    if T ∪ s_i ∈ I, add s_i to T
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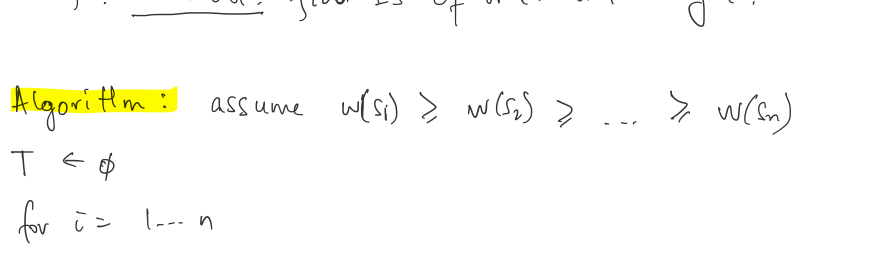
Claim: T obtained by the algorithm is a max wt. IS.

Proof: Let $T = \{t_1, t_2, t_3, \dots, t_k\}$ in decreasing order of wt.



Note: ① Say $X \in \mathcal{I}$, $a \notin X$, $\{a\} \cup X \in \mathcal{I}$. Then for any $Y \subseteq X$, $\{a\} \cup Y \in \mathcal{I}$

② no elt. before t_i extends T_{i-1} (in example above $i = 4$)

Let $T_i = \{t_1, t_2, t_3, \dots, t_i\}$, i.e., T_i is the first i elts of T.

Actually will show that for all $i \leq k$, T_i is the highest wt. IS of size i.

Base case: clearly true for $i = 0$

Suppose T_{i-1} is the max wt. IS of size $i-1$

For a contradiction, assume $w(\hat{T}_i) > w(T_i)$

By exchange property, $\exists a \in \hat{T}_i \setminus T_{i-1}$ s.t. $T_{i-1} \cup \{a\} \in \mathcal{I}$.

But then, a must lie after t_i, or $w(a) \leq w(t_i)$ by Note 2

also, $w(\hat{T}_i \setminus a) \leq w(T_{i-1})$ by Induction hypothesis

Thus $w(\hat{T}_i) \leq w(T_{i-1} \cup t_i) = w(T_i)$ ∎

Do yourself: Now suppose some elements have negative wt. How do you modify algorithm?

OR: if you want max wt. IS of size k, in presence of $-$ve wts.?

OR: if you want min wt. IS?